

# On the asymptotic behavior of symmetric solutions of the Allen-Cahn equation in unbounded domains in $\mathbb{R}^2$

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## Abstract

We consider a Dirichlet problem for the Allen–Cahn equation in a smooth, bounded or unbounded, domain  $\Omega \subset \mathbb{R}^n$ . Under suitable assumptions, we prove an existence result and a uniform exponential estimate for symmetric solutions. In dimension  $n = 2$  an additional asymptotic result is obtained. These results are based on a pointwise estimate obtained for local minimizers of the Allen–Cahn energy.

## 1 Introduction

We consider the Allen-Cahn equation

$$(1.1) \quad \begin{cases} \Delta u = W'(u), & x \in \Omega, \\ u = g, & x \in \partial\Omega. \end{cases}$$

where  $\Omega \subset \mathbb{R}^n$  is a bounded or unbounded domain,  $g : \partial\Omega \rightarrow \mathbb{R}$  is continuous and bounded and  $W : \mathbb{R} \rightarrow \mathbb{R}$  is a  $C^3$  potential.

We are interested in symmetric solutions:

$$u(\hat{x}) = -u(x), \quad \text{for } x \in \Omega$$

where for  $z \in \mathbb{R}^d$  we let  $\hat{z} = (-z_1, z_2, \dots, z_d)$  the reflection in the plane  $z_1 = 0$ . We assume:

**h<sub>1</sub>–**  $W : \mathbb{R} \rightarrow \mathbb{R}$  is an even function:

$$(1.2) \quad W(-u) = W(u), \quad \text{for } u \in \mathbb{R},$$

which has a unique non-degenerate positive minimizer:

$$(1.3) \quad \begin{aligned} 0 = W(1) &< W(u), \quad \text{for } u \geq 0, \\ W''(1) &> 0. \end{aligned}$$

**h<sub>2</sub>–** There is  $M > 0$  such that

$$(1.4) \quad W'(u) \geq 0, \quad \text{for } u \geq M.$$

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**h<sub>3</sub>**—  $\Omega \subset \mathbb{R}^n$  is a domain with nonempty boundary which is symmetric:

$$(1.5) \quad x \in \Omega \Rightarrow \hat{x} \in \Omega,$$

and of class  $C^{2+\alpha}$ . If  $\Omega$  is unbounded we require that  $\Omega$  satisfies a uniform interior sphere condition and that the curvature of  $\partial\Omega$  is bounded in the  $C^\alpha$  sense.

If  $S \subset \mathbb{R}^d$  is a symmetric set, we define  $S^+ := \{x \in S : x_1 > 0\}$ . We first consider the case of general  $n \geq 1$  and prove the existence of a symmetric solution which is near 1 in  $\Omega^+$ . Note that, in general,  $\partial(\Omega^+) \neq (\partial\Omega)^+$ .

**Theorem 1.1.** *Assume that  $W$  and  $\Omega \subset \mathbb{R}^n$  satisfy  $h_1$ ,  $h_2$  and  $h_3$ . Assume that  $g : \partial\Omega \rightarrow \mathbb{R}$  is symmetric and bounded as a  $C^{2,\alpha}(\partial\Omega; \mathbb{R})$  function and satisfies*

$$g(x) \geq 0, \text{ for } x \in (\partial\Omega)^+.$$

*Then, problem (1.1) has a symmetric classical solution  $u \in C^2(\overline{\Omega}; \mathbb{R})$  such that*

$$(1.6) \quad \begin{aligned} u(x) &\geq 0, \text{ for } x \in \Omega^+, \\ |u(x) - 1| &\leq Ke^{-kd(x, \partial(\Omega^+))}, \quad x \in \Omega^+, \end{aligned}$$

*for some positive constants  $k, K$  that depend only on  $W, n$  and on the  $C^1(\overline{\Omega}; \mathbb{R})$  norm of  $u$ .*

(We assume that  $g$  is extended to  $\Omega$  as a symmetric  $C^{2,\alpha}$  map). A similar statement is valid in the case of Neumann boundary conditions.

We then restrict to the case  $n = 2$  and prove the following asymptotic result

**Theorem 1.2.** *Assume  $W$  as in Theorem 1.1 and assume that  $\Omega \subset \mathbb{R}^2$  satisfies  $h_3$  and is convex in  $x_1$  i.e.*

$$(1.7) \quad (x_1, x_2) \in \Omega \Rightarrow (tx_1, x_2) \in \Omega, \text{ for } |t| \leq 1.$$

*Let  $u$  be the solution of the Allen-Cahn equation (1.1) given by Theorem 1.1. Then there exists a continuous decreasing map  $R \rightarrow q(R)$ ,  $\lim_{R \rightarrow +\infty} q(R) = 0$ , such that*

$$(1.8) \quad |u(x_1, x_2) - \bar{u}(x_1)| \leq q(d(x, \partial\Omega)), \text{ for } x \in \Omega,$$

*where  $\bar{u} : \mathbb{R} \rightarrow \mathbb{R}$  is the odd solution of*

$$(1.9) \quad \begin{aligned} v'' &= W'(v), \quad s \in \mathbb{R} \\ \lim_{s \rightarrow +\infty} v(s) &= 1. \end{aligned}$$

*The map  $q$  depends only on  $W, n$  and on the  $C^1(\overline{\Omega}; \mathbb{R})$  norm of  $u$ .*

A convergence result for odd solutions of (1.1) similar to (1.8) valid in the case  $\Omega \subset \mathbb{R}^n$  is a half space was obtained, among other things, in [2] (cfr. Theorem 1.1). The point in Theorem 1.2 is that, even though is restricted to  $n = 2$ , applies to general domains that satisfy (1.7). Some of the ideas in the proof of Theorem 1.2 have been extended and utilized in [1] where the restriction to  $n = 2$  is removed and  $u$  is allowed to be a vector.

The proof of Theorems 1.1 and Theorem 1.2 is variational and is based on a pointwise estimate for *local minimizers* of the Allen-Cahn energy

$$(1.10) \quad J_A(u) := \int_A \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) dx,$$

defined for all bounded domain  $A \subset \mathbb{R}^n$  and  $u \in W^{1,2}(A; \mathbb{R})$ .

**Definition.** Let  $\Omega \subset \mathbb{R}^n$  be a domain. A map  $u \in W_{\text{loc}}^{1,2}(\Omega; \mathbb{R})$  is a local minimizer of the Allen-Cahn energy if

$$(1.11) \quad J_A(u) = \min_{v \in W_0^{1,2}(A; \mathbb{R})} J_A(u + v),$$

for every bounded Lipschitz domain  $A \subset \Omega$ .

In the following we denote by  $k, K$  and  $C$  generic positive constants that can change from line to line.

The pointwise estimate alluded to above is stated in the following

**Theorem 1.3.** *Assume  $W : \mathbb{R} \rightarrow \mathbb{R}$  is a  $C^2$  function such that*

- (i)  $0 = W(0) < W(v)$ , for  $v \in \mathbb{R}$ ,
- (ii)  $\liminf_{|v| \rightarrow +\infty} W(v) > 0$ ,
- (iii)  $W''(0) > 0$ .

*Let  $\Omega \subset \mathbb{R}^n$  a domain and  $u \in C^1(\Omega; \mathbb{R})$  a local minimizer of the Allen-Cahn energy that satisfies*

$$(1.12) \quad |u| + |\nabla u| \leq M_0, \text{ for } x \in \Omega$$

*and some  $M_0 > 0$ .*

*Then there is  $q^* > 0$  with the property that for each  $q \in (0, q^*]$  there is  $R(q) > 0$  such that*

$$(1.13) \quad B_{x, R(q)} \subset \Omega \Rightarrow |u(x)| < q.$$

*Moreover  $R(q)$  can be chosen strictly decreasing and continuous in  $(0, q^*]$ . The inverse map  $q(R)$  satisfies*

$$(1.14) \quad q(R) \leq K e^{-kR}, \quad R \in [R(q^*), +\infty),$$

*for some positive constants  $k, K$  that depend only on  $W, n$  and the bound  $M_0$ .*

The paper is organized as follows. In Sect. 2 we use Theorem 1.3 to prove Theorem 1.1. In Sect. 3 we prove Theorem 1.2. Finally in Sect. 4 we present a proof of Theorem 1.3. The proof is an adaptation to the scalar case of arguments developed in [3] and [4] for the vector Allen-Cahn equation.

## 2 The proof of Theorem 1.1

We first consider the case of  $\Omega$  bounded. Then standard arguments from variational calculus yield the existence of a symmetric minimizer  $u \in g + W_{0,S}^{1,2}(\Omega; \mathbb{R})$  of  $J_\Omega$ ; we denote by  $W_{0,S}^{1,2}(\Omega; \mathbb{R})$  the subspace of symmetric maps of  $W_0^{1,2}(\Omega; \mathbb{R})$ . Let  $g_m = \max_{\partial\Omega} g$ . We can assume

$$(2.1) \quad |u| \leq M' := \max\{M, g_m\},$$

and

$$(2.2) \quad u \geq 0 \text{ on } \Omega^+.$$

To prove (2.1) we let  $v \in g + W_{0,S}^{1,2}(\Omega; \mathbb{R})$  the symmetric function defined by

$$(2.3) \quad v = \begin{cases} u, & \text{on } \Omega^+ \cap \{u \leq M'\}, \\ M', & \text{on } \Omega^+ \cap \{u > M'\}, \end{cases}$$

and observe that if  $\Omega^+ \cap \{u > M'\}$  has positive measure, then  $h_2$  implies

$$(2.4) \quad J_\Omega(u) - J_\Omega(v) = \int_{\Omega^+ \cap \{u > M'\}} (|\nabla u|^2 + 2(W(u) - W(v))) dx > 0,$$

in contradiction with the minimality of  $u$ . The proof of (2.2) is similar.

From the bound (2.1), the smoothness assumption on  $\partial\Omega$  in  $h_3$  and elliptic regularity we obtain that  $u$  is a classical solution of (1.1) and

$$(2.5) \quad \|u\|_{C^{2,\alpha}(\overline{\Omega}; \mathbb{R})} \leq M'',$$

for some constant  $M'' > 0$ . The restriction of  $u$  to  $\Omega^+$  trivially satisfies the definition of minimizer of the Allen-Cahn energy in  $\Omega^+$  with potential  $\tilde{W}$  that, by (2.1) and (2.2), can be identified with any smooth function that satisfies  $\tilde{W}(s) = W(s)$ , for  $s \geq 0$  and  $\tilde{W}(s) > W(|s|)$ , for  $s < 0$ . From this and (2.5) it follows that we can apply Theorem 1.3 to  $\hat{u} = u - 1$  with potential  $\tilde{W}(\cdot + 1)$  and conclude that  $u$  satisfies the exponential estimate

$$(2.6) \quad |u(x) - 1| \leq K e^{-kd(x, \partial\Omega^+)}, \quad \text{for } x \in \Omega^+,$$

with  $k, K$  depending only on  $W$  and  $M''$ . This concludes the proof for  $\Omega$  bounded. If  $\Omega$  is unbounded we consider a sequence of bounded domains  $\Omega_j$ ,  $j \in \mathbb{N}$ , such that  $\Omega_j \subset \Omega_{j+1}$  and  $\Omega = \cup_j \Omega_j$ . From  $h_3$  we can assume that the boundary of  $\Omega_j$  is of class  $C^{2,\alpha}$  and satisfies an interior sphere condition uniformly in  $j \in \mathbb{N}$ . Therefore the same reasoning developed for the case of bounded  $\Omega$  yields

$$(2.7) \quad \|u_j\|_{C^{2,\alpha}(\overline{\Omega_j}; \mathbb{R})} \leq M'', \quad \text{for } j \in \mathbb{N},$$

and

$$(2.8) \quad |u_j(x) - 1| \leq K e^{-kd(x, \partial\Omega_j^+)}, \quad \text{for } x \in \Omega_j^+, j \in \mathbb{N}.$$

The estimate (2.7) implies that, passing to a subsequence if necessary, we can assume that  $u_j$  converges locally in  $C^2$  to a classical solution  $u : \Omega \rightarrow \mathbb{R}$  of (1.1) and (2.8) implies that  $u$  satisfies the exponential estimate in Theorem 1.1. The proof of Theorem 1.1 is complete.

*Remark.* Elliptic regularity implies that we can upgrade the exponential estimate in Theorem 1.1 to

$$(2.9) \quad |u(x) - 1| + |\nabla u(x)| \leq K e^{-kd(x, \partial\Omega^+)}, \quad \text{for } x \in \Omega^+.$$

In the proof of Theorem 1.2 below we make systematic use of the fact that the solution of (1.1) given by Theorem 1.1 is a local minimizer in the sense of Definition 1. This is obvious when  $\Omega$  is a bounded. If  $\Omega$  is unbounded it follows from the fact that  $u = \lim_{j \rightarrow +\infty} u_j$  is the limit of a sequence of minimizers  $u_j : \Omega_j \rightarrow \mathbb{R}$ ,  $\Omega_j$  bounded, that converges to  $u$  uniformly in compacts [7].

### 3 The proof of Theorem 1.2

We divide the proof in several lemmas.

For  $l \in (0, +\infty]$  let

$$(3.1) \quad \mathcal{B}_l := \{ v \in W_{\text{odd}}^{1,2}((-l, l); \mathbb{R}) : v(\pm l) = 0; \|v\|_{1,l} \leq M'' \},$$

where  $W_{\text{odd}}^{1,2}((-l, l); \mathbb{R}) \subset W^{1,2}((-l, l); \mathbb{R})$  is the subset of the odd functions and  $\|v\|_{1,l}$  is the usual  $W^{1,2}$  norm of  $v$ . Let  $\mathcal{S} \subset W_{\text{odd}}^{1,2}((-l, l); \mathbb{R})$  be defined by

$$(3.2) \quad \mathcal{S} := \{ \nu \in W_{\text{odd}}^{1,2}((-l, l); \mathbb{R}) : \|\nu\|_l = 1 \},$$

where, for  $l \in (0, +\infty]$ ,  $\|v\|_l$  denotes the  $L^2((-l, l); \mathbb{R})$  norm of  $v$ . In particular  $\|v\|_\infty = \|v\|_{L^2(\mathbb{R})}$ . For  $v \in \mathcal{B}_l$  we write

$$v = q\nu, \quad \text{with } q = \|v\|_l \text{ and } \nu \in \mathcal{S}.$$

For  $w \in W^{1,2}((-l, l); \mathbb{R})$  we set

$$(3.3) \quad e_l(w) = \int_{-l}^l \left( \frac{1}{2} |w_{x_1}|^2 + W(w) \right) dx_1.$$

Define  $E_l : \mathcal{B}_l \rightarrow \mathbb{R}$  by setting

$$(3.4) \quad \begin{aligned} E_l(v) &= e_l(\bar{u} + v) - e_l(\bar{u}) \\ &= \frac{1}{2} \int_{-l}^l (|\bar{u}_{x_1} + v_{x_1}|^2 - |\bar{u}_{x_1}|^2) dx_1 + \int_{-l}^l [W(\bar{u} + v) - W(\bar{u})] dx_1. \end{aligned}$$

**Lemma 3.1.** *There exist  $q_0 > 0, c > 0$ , independent of  $l > 1$ , such that for  $v \in \mathcal{B}_l$ ,  $v = q\nu$ , we have*

$$(3.5) \quad E_l(q\nu) \geq \frac{1}{2} c^2 q^2, \quad 0 < q \leq q_0, \quad \nu \in \mathcal{S},$$

$$(3.6) \quad E_l(q\nu) \geq \frac{1}{2} c^2 q_0^2, \quad q_0 \leq q, \quad \nu \in \mathcal{S}.$$

Moreover it results

$$(3.7) \quad D_{qq} E_l(q\nu) \geq c^2, \quad 0 < q \leq q_0, \quad \nu \in \mathcal{S}.$$

*Proof.* From (1.9) and  $v(\pm l) = 0$  it follows

$$(3.8) \quad \int_{-l}^l (\bar{u}_{x_1} v_{x_1} + W'(\bar{u})v) dx_1 = \int_{-l}^l (-\bar{u}_{x_1 x_1} + W'(\bar{u}))v dx_1 = 0.$$

Therefore, for  $v \in \mathcal{B}_l$ , we can rewrite  $E_l(v)$  in the form

$$(3.9) \quad \begin{aligned} E_l(v) &= \int_{-l}^l \left( \frac{1}{2} W''(\bar{u})v^2 + \frac{v_{x_1}^2}{2} \right) dx_1 \\ &\quad + \int_{-l}^l \left[ W(\bar{u} + v) - W(\bar{u}) - W'(\bar{u})v - \frac{1}{2} W''(\bar{u})v^2 \right] dx_1, \end{aligned}$$

where we have also added and subtracted  $\frac{1}{2}W''(\bar{u})v^2$ .

By differentiating (1.9) we see that  $\bar{u}_{x_1}$  is an eigenfunction corresponding to the eigenvalue  $\lambda = 0$  for the operator  $L$  defined by

$$L : W^{1,2}(\mathbb{R}) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}),$$

$$Lw := -w_{x_1 x_1} + W''(\bar{u})w.$$

Since  $\bar{u}$  is increasing and odd,  $\bar{u}_{x_1}$  is positive and even. On the other hand the assumption  $W''(\pm 1) > 0$  implies, see e.g. Theorem A.2 of [6] (pag. 140), that the essential spectrum of  $L$  is contained in  $[a, +\infty)$  for some  $a > 0$ . Therefore, if we restrict to the subset of odd functions we can conclude that there exists a positive constant  $c_1$  such that

$$(3.10) \quad \int_{-\infty}^{+\infty} \left( \frac{1}{2}W''(\bar{u})\phi^2 + \frac{\phi_{x_1}^2}{2} \right) dx_1 = \frac{1}{2} \int_{-\infty}^{+\infty} (L\phi)\phi dx_1 \geq c_1^2 \int_{-\infty}^{+\infty} \phi^2 dx_1, \quad \text{for all } \phi \in W_{\text{odd}}^{1,2}(\mathbb{R}).$$

In particular, given  $v \in \mathcal{B}_l$ , we can apply (3.10) to the trivial extension  $\tilde{v}$  of  $v$  to obtain

$$(3.11) \quad \int_{-l}^{+l} \left( \frac{1}{2}W''(\bar{u})v^2 + \frac{v_{x_1}^2}{2} \right) dx_1 \geq c_1^2 \int_{-l}^{+l} v^2 dx_1, \quad \text{for all } v \in \mathcal{B}_l.$$

Since  $v \in \mathcal{B}_l$  implies  $v(-l) = 0$ , we have  $v^2(x_1) = 2 \int_{-l}^{x_1} v(s)v_{x_1}(s)ds$  and therefore

$$(3.12) \quad \|v\|_{L^\infty(-l,l)} \leq \sqrt{2}\|v\|_l^{1/2}\|v\|_{1,l}^{1/2} \leq C\|v\|_l^{1/2}, \quad \text{for } v \in \mathcal{B}_l,$$

with  $C = \sqrt{2M''}$ . Fix  $q_0 > 0$  and let  $\overline{W}''' = \max_{|s| \leq 1+Cq_0^{1/2}} |W'''(s)|$ . Then, for some map  $x_1 \rightarrow \theta(x_1) \in (0, 1)$ , we have

$$(3.13) \quad \left| \int_{-l}^l (W(\bar{u} + v) - W(\bar{u}) - W'(\bar{u})v - \frac{1}{2}W''(\bar{u})v^2) dx_1 \right| = \left| \int_{-l}^l W'''(\bar{u} + \theta v) \frac{v^3}{6} dx_1 \right| \\ \leq \frac{1}{6} C \overline{W}''' q_0^{1/2} \int_{-l}^l v^2 dx_1, \quad \text{for } \|v\|_l \leq q_0.$$

From (3.9), (3.11) and (3.13), if we choose  $q_0 > 0$  so small that  $C\overline{W}''' q_0^{1/2} \leq 3c_1^2$ , it follows

$$E_l(qv) = E_l(v) \geq \frac{1}{2}c_1^2 \int_{-l}^l v^2 dx_1, \quad \text{for } v \in \mathcal{B}_l, \quad 0 < q = \|v\|_l \leq q_0,$$

that is (3.5).

To show (3.6) let us consider the minimization problem

$$(3.14) \quad \min_{\substack{v \in \mathcal{B}_l \\ \|v\|_l \geq q_0}} E_l(v).$$

It is easy to construct a smooth odd map  $w \in \mathcal{B}_l$  that satisfies the constraint  $\|w\|_l \geq q_0$ . Therefore there exists a minimizing sequence  $\{v_j\} \subset \mathcal{B}_l$  that satisfies  $\|v_j\|_l \geq q_0$ ,  $j \in \mathbb{N}$ , and

$$(3.15) \quad E_l(v_j) \leq E_l(w), \quad j \in \mathbb{N}.$$

From (3.15) and standard arguments from variational calculus it follows that there is  $\bar{v}_l \in \mathcal{B}_l$  and a subsequence  $\{v_{j_h}\}$  such that

$$(3.16) \quad \begin{aligned} \liminf_{h \rightarrow +\infty} E_l(v_{j_h}) &\geq E_l(\bar{v}_l), \\ \lim_{h \rightarrow +\infty} \|v_{j_h} - \bar{v}_l\|_l &= 0. \end{aligned}$$

It follows  $\|\bar{v}_l\|_l \geq q_0$  and  $\bar{v}_l$  is a minimizer of (3.14). Since  $E_l(0) = 0$  and  $v = 0$  is the unique minimizer of  $E_l$  on  $\mathcal{B}_l$ , this implies  $E_l(\bar{v}_l) = \alpha_l > 0$ , and therefore

$$(3.17) \quad E_l(q\nu) \geq \alpha_l, \quad \text{for } q \geq q_0.$$

Note that  $\alpha_l$  is non increasing with  $l$ . Indeed, if  $l_1 < l_2$  and  $v \in \mathcal{B}_{l_1}$ , then the trivial extension  $\tilde{v}$  of  $v$  to  $[-l_2, l_2]$  satisfies  $E_{l_2}(\tilde{v}) = E_{l_1}(v)$  and belongs to  $\mathcal{B}_{l_2}$ . Therefore, there exists  $\lim_{l \rightarrow +\infty} \alpha_l$ . We claim that

$$(3.18) \quad \lim_{l \rightarrow +\infty} \alpha_l = \alpha > 0.$$

Let  $\{l_k\}_k$  be a sequence of positive numbers such that  $l_k \rightarrow +\infty$  for  $k \rightarrow +\infty$ . Let  $\bar{v}_k$  be a minimizer of problem (3.14) for  $l = l_k$  and let  $\tilde{\bar{v}}_k : \mathbb{R} \rightarrow \mathbb{R}$  be the trivial extension of  $\bar{v}_k$ , we may assume that the sequence  $\{\tilde{\bar{v}}_k\}_k$  converges in  $L^2(\mathbb{R})$  and weakly in  $W^{1,2}(\mathbb{R})$  to a map  $\bar{v}$  which satisfies, by lower semicontinuity of  $E_\infty$ ,  $\|\bar{v}\|_\infty \geq q_0$  and  $\alpha \geq E_\infty(\bar{v})$ . Since  $v = 0$  is the unique minimizer of  $E_\infty$ , this implies  $\alpha \geq E_\infty(\bar{v}) > 0$  and proves (3.18). From (3.17) we then deduce

$$E_l(q\nu) \geq \alpha = \frac{1}{2}c_2^2q_0^2,$$

for a suitable constant  $c_2$  independent of  $l$ . Then both (3.5) and (3.6) hold with  $c := \min\{c_1, c_2\}$ .

To prove (3.7) we note that setting  $v = q\nu$  in (3.9) yields

$$(3.19) \quad \begin{aligned} E_l(q\nu) &= q^2 \int_{-l}^l \left( \frac{1}{2}W''(\bar{u})\nu^2 + \frac{\nu_{x_1}^2}{2} \right) dx_1 \\ &\quad + \int_{-l}^l \left[ W(\bar{u} + q\nu) - W(\bar{u}) - qW'(\bar{u})\nu - \frac{1}{2}q^2W''(\bar{u})\nu^2 \right] dx_1, \end{aligned}$$

which via (3.11) implies

$$D_{qq}E_l(q\nu)|_{q=0} = \int_{-l}^l \left( W''(\bar{u})\nu^2 + \nu_{x_1}^2 \right) dx_1 \geq 2c_1^2.$$

This concludes the proof.  $\square$

For  $r > 0, l > 0$  and  $\eta \in \mathbb{R}$ , we denote by  $C_l^r(\eta)$  the set

$$(3.20) \quad C_l^r(\eta) := \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < l, |x_2 - \eta| < r\}.$$

In the following, whenever possible, we assume that by a translation we can reduce to the case  $\eta = 0$  and write simply  $C_l^r$  instead of  $C_l^r(0)$ .

The introduction of the map  $E_l$  allows to represent the energy  $J_{C_l^r}(v)$  of an odd map  $v : C_l^r \rightarrow \mathbb{R}$  that satisfies  $v(x) = \bar{u}(x_1)$ , for  $|x_1| < l, |x_2| < r$  in a particular form that we now derive. We have

$$(3.21) \quad J_{C_l^r}(v) = \frac{1}{2} \int_{-r}^r \int_{-l}^l |v_{x_2}|^2 dx_1 dx_2 + \int_{-r}^r E_l(v - \bar{u}) dx_2 + \int_{-r}^r e_l(\bar{u}) dx_2.$$

If we set

$$q^v(x_2) := \|v(\cdot, x_2) - \bar{u}(\cdot)\|_l > 0$$

then  $v - \bar{u} = q^v \nu^v$  with  $\nu^v = \frac{v - \bar{u}}{\|v - \bar{u}\|_l}$ . We observe that  $v_{x_2} = q_{x_2}^v \nu^v + q^v \nu_{x_2}^v$  implies

$$(3.22) \quad \int_{-l}^l |v_{x_2}|^2 dx_1 = |q_{x_2}^v|^2 + (q^v)^2 \int_{-l}^l |\nu_{x_2}^v|^2 dx_1,$$

where we have also used  $\int_{-l}^l \nu_{x_2}^v \nu^v dx_1 = 0$ . It follows

$$(3.23) \quad J_{C_l^r}(v) = \frac{1}{2} \int_{-r}^r (|q_{x_2}^v|^2 + (q^v)^2 \|\nu_{x_2}^v\|_l^2) dx_2 + \int_{-r}^r E_l(q^v \nu^v) dx_2 + \int_{-r}^r e_l(\bar{u}) dx_2.$$

Assume now that  $w : C_l^r \rightarrow \mathbb{R}$  is of the form

$$w(x_1, x_2) = \bar{u}(x_1) + q^w(x_2) \nu^v(x_1, x_2);$$

then we have

$$(3.24) \quad J_{C_l^r}(w) = \frac{1}{2} \int_{-r}^r (|q_{x_2}^w|^2 + (q^w)^2 \|\nu_{x_2}^v\|_l^2) dx_2 + \int_{-r}^r E_l(q^w \nu^v) dx_2 + \int_{-r}^r e_l(\bar{u}) dx_2.$$

For later reference we state

**Lemma 3.2.** *Let  $f : [-l, l] \rightarrow \mathbb{R}$  be a Lipschitz continuous function satisfying*

$$(3.25) \quad |f(s)| + |f'(s)| \leq K e^{-k|s|}, \quad \text{for } s \in (-l, l).$$

*Then, there is a constant  $C_2 > 0$  independent of  $l \geq 1$  such that*

$$(3.26) \quad \|f\|_{L^\infty} \leq C_2 \|f\|_l^{\frac{2}{3}}$$

*Proof.* From (3.25) there is  $\bar{s} \in [-l, l]$  such that  $|f(s)| \leq m := |f(\bar{s})|$ ,  $s \in [-l, l]$ . From this and  $|f'(s)| \leq K$  it follows

$$|f(s)| \geq m - K|s - \bar{s}|, \quad \text{for } s \in [-l, l] \cap [\bar{s} - m/K, \bar{s} + m/K]$$

and a simple computation gives (3.26).  $\square$

**Lemma 3.3.** *There exist  $J_0 > 0$ ,  $C > 0$ ,  $k > 0$  and a map  $(0, \infty) \ni r \rightarrow l_r > 0$  such that, given  $r > 0$ , if  $l \geq l_r$  and*

$$(3.27) \quad C_l^r \subset \Omega, \quad d(C_l^r, \partial\Omega) > l,$$

*then there is a Lipschitz continuous function  $v$  with the following properties:*

- (i)  $v(x) = \bar{u}(x_1)$ , for  $x \in \partial C_{l+\delta/2}^{r+\delta/2}$ ;
- (ii)  $v(x) = u(x)$ , for  $x \in \bar{C}_{l,r}$  and  $x \in \Omega \setminus C_{l+\delta}^{r+\delta}$ ;
- (iii)  $\|v(\cdot, x_2) - u(\cdot, x_2)\|_{l+\delta/2} \leq C e^{-kl}$ , for  $x_2 \in [-r, r]$ ,
- (iv)  $J_{\overline{C_{l+\delta}^{r+\delta} \setminus C_l^r}}(v) - J_{\overline{C_{l+\delta}^{r+\delta} \setminus C_l^r}}(u) \leq J_0$ ,

*where  $\delta > 0$  is a fixed constant.*



*Proof.* We set

$$(3.28) \quad v = u \quad \text{for } x \in \overline{C_l^r} \cup (\Omega \setminus C_{l+\delta}^{r+\delta}).$$

To define  $v$  in  $C_{l+\delta}^{r+\delta} \setminus \overline{C_l^r}$  let  $S_1 \subset \mathbb{R}^2$  be the sector  $S_1 = \{x : x_1 \geq l - r, |x_2| < x_1 - l + r\}$  and let  $(\rho, \theta)$  polar coordinates in  $S_1$  with origin in the vertex  $(l - r, 0)$  of  $S_1$  and polar axis parallel to  $x_1$ . We let  $x(\rho, \theta)$  denote the point of  $S_1$  with polar coordinates  $(\rho, \theta)$ . We define  $v$  in the trapezoid  $T_1 := (C_{l+\delta}^{r+\delta} \setminus \overline{C_l^r}) \cap S_1$  by setting

$$(3.29) \quad v(x(\rho, \theta)) := \left(1 - \left|1 - 2 \frac{\rho - \rho_1(\theta)}{\rho_2(\theta) - \rho_1(\theta)}\right|\right) \overline{u}(l + \delta/2) + \left|1 - 2 \frac{\rho - \rho_1(\theta)}{\rho_2(\theta) - \rho_1(\theta)}\right| u(x(\rho, \theta)),$$

$$\text{for } \rho \in (\rho_1(\theta), \rho_2(\theta)), |\theta| \leq \frac{\pi}{4},$$

where  $\rho_1(\theta)$ , and  $\rho_2(\theta)$  are defined by the conditions  $x_1(\rho_1(\theta), \theta) = l$  and  $x_1(\rho_2(\theta), \theta) = l + \delta$ .

In the trapezoid  $T_2 := (C_{l+\delta}^{r+\delta} \setminus \overline{C_l^r}) \cap S_2$ ,  $S_2 = \{x : x_2 \geq r - l, |x_1| < x_2 - r + l\}$  we define

$$(3.30) \quad v(x(\varrho, \phi)) := \left(1 - \left|1 - 2 \frac{\varrho - \varrho_1(\phi)}{\varrho_2(\phi) - \varrho_1(\phi)}\right|\right) \overline{u}(x_1(\frac{\varrho_1(\phi) + \varrho_2(\phi)}{2}, \phi))$$

$$+ \left|1 - 2 \frac{\varrho - \varrho_1(\phi)}{\varrho_2(\phi) - \varrho_1(\phi)}\right| u(x(\varrho, \phi)), \quad \text{for } \varrho \in (\varrho_1(\phi), \varrho_2(\phi)), |\phi| \leq \frac{\pi}{4},$$

where  $(\varrho, \phi)$  are polar coordinates in  $S_2$  and  $\varrho_1(\phi)$ , and  $\varrho_2(\phi)$  are defined by the conditions  $x_2(\varrho_1(\phi), \phi) = r$  and  $x_2(\varrho_2(\phi), \phi) = r + \delta$ . In the remaining two trapezoids we define  $v$  in a similar way.

The maps defined by (3.28), (3.29) and (3.30) are Lipschitz continuous in the closure of their domains of definition and join continuously on the boundary of  $C_{l+\delta}^{r+\delta} \setminus \overline{C_l^r}$  and along the line  $\theta = \pi/4$ . Indeed (3.29) and (3.30) yield

$$\begin{aligned} \rho = \rho_i(\theta) &\Rightarrow v(x(\rho_i(\theta), \theta)) = u(x(\rho_i(\theta), \theta)), \\ \varrho = \varrho_i(\phi) &\Rightarrow v(x(\varrho_i(\phi), \phi)) = u(x(\varrho_i(\phi), \phi)) \end{aligned} \quad i = 1, 2.$$

and

$$\begin{aligned} x(s + \rho_1(\pi/4), \pi/4) &= x(s + \varrho_1(\pi/4), \pi/4), \\ \Rightarrow v(x(s + \rho_1(\pi/4), \pi/4)) &= v(x(s + \varrho_1(\pi/4), \pi/4)), \quad s \in [0, \sqrt{2}\delta]. \end{aligned}$$

Therefore we conclude that, as defined, the map  $v$  is uniformly Lipschitz continuous on  $\Omega$ . The fact that  $v$  satisfies (i) follows from (3.29) and (3.30) that imply

$$\begin{aligned} \rho = (\rho_1(\theta) + \rho_2(\theta))/2 &\Rightarrow v(x((\rho_1(\theta) + \rho_2(\theta))/2, \theta)) = \overline{u}(l + \delta/2), \\ \varrho = (\varrho_1(\phi) + \varrho_2(\phi))/2 &\Rightarrow v(x((\varrho_1(\phi) + \varrho_2(\phi))/2, \phi)) = \overline{u}(x_1((\varrho_1(\phi) + \varrho_2(\phi))/2)). \end{aligned}$$

To prove (iii) and (iv) we use the estimate

$$(3.31) \quad |\overline{u}(s) - 1| + |\overline{u}'(s)| \leq K e^{-ks}, \quad \text{for } s \geq 0,$$

and the estimate for the solution  $u$  established in (2.9). Set  $\lambda := \left| 1 - 2 \frac{\rho - \rho_1(\theta)}{\rho_2(\theta) - \rho_1(\theta)} \right| \in [0, 1]$ , then (3.29) implies

$$(3.32) \quad v - 1 = (1 - \lambda)(\bar{u}(l + \delta/2) - 1) + \lambda(u(x(\rho, \theta)) - 1).$$

This,  $x_1(\rho, \theta) > l$  on  $T_1$ , (3.31) and (2.9) imply  $|v - 1| \leq Ke^{-kl}$  on  $T_1$  and therefore (iii) follows. Moreover, since  $W(s) = O((s - 1)^2)$  for  $s - 1$  small, it results

$$(3.33) \quad \int_{T_1} (W(v) - W(u)) dx \leq \int_{T_1} W(v) dx \leq Cr\delta e^{-\gamma l},$$

where  $\gamma, C$  denote a generic positive constants independent of  $r$  and  $l$ . Differentiating (3.29) in  $x$  yields

$$(3.34) \quad \nabla v = (1 - \lambda)\bar{u}'(l + \delta/2)e_1 + \lambda \nabla u(x(\rho, \theta)) - (\bar{u}(l + \delta/2) - u(x(\rho, \theta)))\nabla \lambda,$$

where  $e_i$ ,  $i = 1, 2$  is the standard basis of  $\mathbb{R}^2$ . Since  $\nabla \lambda$  is bounded on  $T_1$  with a bound independent of  $r$  and  $l$ , using again (3.31) and (2.9) we see that (3.34) implies

$$(3.35) \quad \int_{T_1} \frac{1}{2} (|\nabla v|^2 - |\nabla u|^2) dx \leq \int_{T_1} \frac{1}{2} |\nabla v|^2 dx \leq Cr\delta e^{-\gamma l}.$$

To estimate  $J_{T_2}(v) - J_{T_2}(u)$  we proceed in a similar way. We set  $\lambda = \left| 1 - 2 \frac{\varrho - \varrho_1(\phi)}{\varrho_2(\phi) - \varrho_1(\phi)} \right|$  and write equations analogous to (3.32) and (3.34). From these equations, using as before the estimates (3.31) and (2.9), and observing that

$$(3.36) \quad \varrho \in (\varrho_1(\phi), \varrho_2(\phi)) \Rightarrow |x_1(\varrho, \phi)| \geq |x_1(\varrho_1(\phi), \phi)| = l |\tan \phi|,$$

it follows that there is a constant  $C_0$  independent of  $r$  and  $l$  such that  $J_{T_2}(v) - J_{T_2}(u) \leq J_{T_2}(v) \leq C_0$ . This, (3.33) and (3.35) imply

$$(3.37) \quad J_{\overline{C_{l+\delta}^{r+\delta} \setminus C_l^r}}(v) - J_{\overline{C_{l+\delta}^{r+\delta} \setminus C_l^r}}(u) \leq J_{\overline{C_{l+\delta}^{r+\delta} \setminus C_l^r}}(v) \leq 2(C_0 + Cr\delta e^{-\gamma l})$$

and (iv) follows with  $J_0 = 4C_0$  and  $l_r = -\frac{1}{\gamma} \ln \frac{Cr\delta}{C_0}$ .  $\square$

Arguments analogous to the ones in the proof of Lemma 3.3 prove

**Lemma 3.4.** *Assume that  $C_l^r$  satisfies (3.27). Then there is a Lipschitz continuous function  $v$  with the following properties:*

- (i)  $v(x) = \bar{u}(x_1)$ , for  $x \in \{-l - \delta/2, l + \delta/2\} \times [-r - \delta/2, r + \delta/2]$ ,
- (ii)  $v(x) = u(x)$ , for  $x \in \Omega \setminus ((-l - \delta, -l) \cup (l, l + \delta)) \times [-r - \delta, r + \delta]$ ,
- (iii)  $\|v(\cdot, x_2) - u(\cdot, x_2)\|_{l+\delta/2} \leq Ce^{-\gamma l}$ , for  $x_2 \in [-r - \delta/2, r + \delta/2]$ ,
- (iv)  $J(v) - J(u) \leq Cre^{-\gamma l}$ ,

for some constants  $C, \gamma > 0$ .

**Lemma 3.5.** *Let  $q_0$  and  $c$  be as in Lemma 3.1. Given  $\bar{q} < q_0$ , fix  $r > 0$  such that*

$$(3.38) \quad \frac{1}{2}c^2\bar{q}^2r > 8J_0,$$

where  $J_0$  is the constant in (iv) in Lemma 3.3. There is  $l(\bar{q}) > 0$  such that, provided (3.27) is satisfied with  $l \geq \max\{l_r, l(\bar{q})\}$ , then there exist  $a_- \in (-r, -r/2)$  and  $a_+ \in (r/2, r)$  such that

$$(3.39) \quad \|u(\cdot, a_\pm) - \bar{u}\|_{l+\delta/2} < \bar{q}.$$

*Proof.* Let  $v$  be the map constructed in Lemma 3.3. For each  $\eta \in [-r, r/2]$  let  $\mathcal{A}_\eta \subset \mathbb{R}$  be the set

$$(3.40) \quad \mathcal{A}_\eta := \left\{x_2 \in (\eta, \eta + r/2) : q^v(x_2) = \|v(\cdot, x_2) - \bar{u}\|_{l+\delta/2} \geq \frac{\bar{q}}{2}\right\}.$$

Then, we have

$$(3.41) \quad J_{C_{l+\delta/2}^{r+\delta/2}}(v) - J_{C_{l+\delta/2}^{r+\delta/2}}(\hat{v}) \geq |\mathcal{A}_\eta| \frac{1}{2}c^2\frac{\bar{q}^2}{4}, \quad \text{for } \eta \in [-r, r/2],$$

where  $\hat{v}$  be the function that coincides with  $v$  outside  $C_{l+\delta/2}^{r+\delta/2}$  and with  $\bar{u}$  inside  $C_{l+\delta/2}^{r+\delta/2}$ . Note that, since  $v$  coincides with  $\bar{u}$  on the boundary of  $C_{l+\delta/2}^{r+\delta/2}$ ,  $\hat{v}$  is a Lipschitz map. To prove (3.41), we observe that from the definition of  $\hat{v}$  and of  $E_l$  in (3.4) we have (with  $w = v - \bar{u}$ )

$$\begin{aligned} J_{C_{l+\delta/2}^{r+\delta/2}}(v) - J_{C_{l+\delta/2}^{r+\delta/2}}(\hat{v}) &= \frac{1}{2} \int_{C_{l+\delta/2}^{r+\delta/2}} |w_{x_2}|^2 dx_1 dx_2 + \int_{-r-\delta/2}^{r+\delta/2} E_{l+\delta/2}(w) dx_2 \\ &\geq \int_{-r}^r E_{l+\delta/2}(w) dx_2 \geq \frac{1}{2} |\mathcal{A}_\eta| c^2 \frac{\bar{q}^2}{4}, \quad \text{for } \eta \in [-r, r/2] \end{aligned}$$

where we have also used (3.40) and (3.5), (3.6) in Lemma 3.1. Then, from Lemma 3.3 and (3.41), it follows

$$(3.42) \quad 0 \geq J_{C_{l+\delta/2}^{r+\delta/2}}(v) - J_{C_{l+\delta/2}^{r+\delta/2}}(\hat{v}) \geq |\mathcal{A}_\eta| \frac{1}{2}c^2\frac{\bar{q}^2}{4} - J_0 > (|\mathcal{A}_\eta| - \frac{r}{2}) \frac{1}{2}c^2\frac{\bar{q}^2}{4}, \quad \text{for } \eta \in [-r, r/2]$$

and therefore

$$(3.43) \quad |\mathcal{A}_\eta| < \frac{r}{2}, \quad \text{for } \eta \in [-r, r/2].$$

This inequality and the definition (3.40) of  $\mathcal{A}_\eta$  imply the existence of  $a_- \in (-r, -r/2) \setminus \mathcal{A}_0$  and  $a_+ \in (r/2, r) \setminus \mathcal{A}_{3r/2}$  such that

$$(3.44) \quad \|v(\cdot, a_\pm) - \bar{u}\|_{l+\delta/2} < \frac{\bar{q}}{2}.$$

This and (iii) in Lemma 3.3 imply (3.39) provided  $l \geq l(\bar{q}) := \frac{1}{k} \ln \frac{2C}{\bar{q}}$ .  $\square$

**Lemma 3.6.** *Given  $\epsilon > 0$  there is  $l_\epsilon$  such that*

$$x \in \Omega, \quad d(x, \partial\Omega) \geq l_\epsilon \Rightarrow |u(x) - \bar{u}(x_1)| \leq \epsilon.$$

*Proof.* Set  $d_\epsilon := \frac{1}{k} \ln \frac{2K}{\epsilon}$  and assume that  $d(x, \partial\Omega^+) \geq d_\epsilon$ . Then (1.6)<sub>2</sub> and (3.31) imply

$$|u(x) - \bar{u}(x_1)| \leq |u(x) - 1| + |1 - \bar{u}(x_1)| \leq \epsilon.$$

This and the oddness of  $u$  imply that it suffices to consider the points  $x \in \Omega^+$  which have  $d(x, \partial\Omega) \geq d_\epsilon$  and  $x_1 \in [0, d_\epsilon]$ . Assume  $\tilde{x} \in \Omega^+$  is a point with these properties that satisfies  $|u(\tilde{x}) - \bar{u}(\tilde{x}_1)| > \epsilon$ . Then from (2.5) and (3.31) it follows

$$(3.45) \quad |u(\tilde{x}) - \bar{u}(\tilde{x}_1)| - |u(x) - \bar{u}(x_1)| \leq 2\mu(|x_1 - \tilde{x}_1| + |x_2 - \tilde{x}_2|),$$

where  $\mu := \max\{M'', K\}$ . Then,

$$(3.46) \quad |u(x) - \bar{u}(x_1)| > \frac{\epsilon}{2}, \quad \text{for } |x_1 - \tilde{x}_1| < \epsilon/8\mu, \quad |x_2 - \tilde{x}_2| < \epsilon/8\mu.$$

From this inequality it follows

$$(3.47) \quad \|u(\cdot, x_2) - \bar{u}\|_{l+\delta/2} \geq \frac{1}{4\sqrt{\mu}} \epsilon^{\frac{3}{2}}, \quad \text{for } |x_2 - \tilde{x}_2| < \epsilon/8\mu$$

and thus, recalling Lemma 3.4 (iii)

$$(3.48) \quad \|v(\cdot, x_2) - \bar{u}\|_{l+\delta/2} \geq \frac{1}{8\sqrt{\mu}} \epsilon^{\frac{3}{2}}, \quad \text{for } |x_2 - \tilde{x}_2| < \epsilon/8\mu.$$

Set  $q^* := \frac{1}{4\sqrt{\mu}} \epsilon^{\frac{3}{2}}$  and  $\bar{q} = q^*/N$  where  $N > 0$  is a fixed number to be chosen later. In the remaining part of the proof we consider a certain number of lower bounds for  $l$  and we always assume that (3.27) is satisfied for  $l > l_M$  where  $l_M$  represents the maximum of the values  $l_r, l(\bar{q}), \dots$  introduced up to the point considered in the proof.

From Lemma 3.5, if  $N$  is such that  $\bar{q} < q_0$ , there is  $r$  such that, for  $l$  sufficiently large, there exist  $a_- \in (\tilde{x}_2 - r, \tilde{x}_2 - r/2)$  and  $a_+ \in (\tilde{x}_2 + r/2, \tilde{x}_2 + r)$  with the property

$$(3.49) \quad \|u(\cdot, a_\pm) - \bar{u}\|_{l+\delta/2} < \bar{q}.$$

Moreover, from Lemma 3.4, for  $l$  sufficiently large, the map  $v$  defined in the lemma satisfies  $\|u(\cdot, a_\pm) - v(\cdot, a_\pm)\|_{l+\delta/2} < \bar{q}$  and therefore we have

$$(3.50) \quad \|v(\cdot, a_\pm) - \bar{u}\|_{l+\delta/2} < 2\bar{q}.$$

Let  $Q := (-l - \delta/2, l + \delta/2) \times (a_-, a_+)$  and let  $w$  the map defined by

$$(3.51) \quad w = \begin{cases} v, & \text{on } \Omega \setminus Q, \\ v, & \text{on } (-l - \delta/2, l + \delta/2) \times \{x_2\}, \quad x_2 \in (a_-, a_+) \\ & \text{if } q^v(x_2) \leq 2\bar{q}, \\ \bar{u} + 2\bar{q}v, & \text{on } (-l - \delta/2, l + \delta/2) \times \{x_2\}, \quad x_2 \in (a_-, a_+) \\ & \text{if } q^v(x_2) > 2\bar{q}. \end{cases}$$

This definition implies in particular

$$\|w(\cdot, \tilde{x}_2) - \bar{u}\|_{l+\delta/2} \leq 2\bar{q} = \frac{2}{N} \frac{1}{4\sqrt{\mu}} \epsilon^{\frac{3}{2}}.$$

Then Lemma 3.2, provided  $N$  is chosen sufficiently large, implies

$$(3.52) \quad |w(\tilde{x}) - \bar{u}(\tilde{x}_1)| \leq C_2 \left( \frac{2}{N} \frac{1}{4\sqrt{\mu}} \right)^{\frac{2}{3}} \epsilon < \epsilon.$$

On the other hand (3.51), (3.23) and (3.24) imply

(3.53)

$$\begin{aligned}
J_Q(v) - J_Q(w) &= \int_{\{q^v > 2\bar{q}\}} \left[ \frac{1}{2}(|q_{x_2}^v|^2 + ((q^v)^2 - 4\bar{q}^2)\|\nu^v\|_{l+\delta/2}^2) \right. \\
&\quad \left. + E_{l+\delta/2}(q^v \nu^v) - E_{l+\delta/2}(2\bar{q} \nu^v) \right] dx_2 \\
&\geq \int_{\{q^v > 2\bar{q}\}} [E_{l+\delta/2}(q^v \nu^v) - E_{l+\delta/2}(2\bar{q} \nu^v)] dx_2 \\
&\geq \int_{\{q^v > q^*\}} [E_{l+\delta/2}(q^* \nu^v) - E_{l+\delta/2}(2\bar{q} \nu^v)] dx_2.
\end{aligned}$$

From (3.7), for  $q \leq q_0$ , we have  $D_q E_l(q\nu) \geq c^2 q$  and therefore, recalling also that  $\bar{q} = q^*/N$ , we have

$$(3.54) \quad E_{l+\delta/2}(q^* \nu^v) - E_{l+\delta/2}(2\bar{q} \nu^v) \geq \frac{1}{2} c^2 (q^*)^2 \left(1 - \frac{4}{N^2}\right)$$

which via (3.48) yields

$$\int_{\{q^v > q^*\}} [E_{l+\delta/2}(q^* \nu^v) - E_{l+\delta/2}(2\bar{q} \nu^v)] dx_2 \geq \frac{1}{2} c^2 (q^*)^2 \left(1 - \frac{4}{N^2}\right) \frac{\epsilon}{4\mu}.$$

Then, from (3.53) and  $q^* = \frac{1}{4\sqrt{\mu}} \epsilon^{\frac{3}{2}}$  we obtain

$$J_Q(v) - J_Q(w) \geq \frac{c^2}{128\mu^2} \left(1 - \frac{4}{N^2}\right) \epsilon^4.$$

From this and Lemma 3.4 (iv) it follows

$$(3.55) \quad J_Q(u) - J_Q(w) = J_Q(u) - J_Q(v) + J_Q(v) - J_Q(w) > 0,$$

provided  $l$  satisfies, beside previous lower bounds,  $l > l^*$  where  $l^*$  is defined by the condition  $C r e^{-\gamma l^*} = \frac{c^2}{128\mu^2} \left(1 - \frac{4}{N^2}\right) \epsilon^4$ . From the above part of the proof it follows that, if we set  $l_\epsilon = 2l_M$  and if  $\tilde{x}$  is such that  $d(\tilde{x}, \partial\Omega) \geq l_\epsilon$ , then we can construct as before the set  $Q$  and the map  $w$  that coincides with  $u$  outside  $Q$  and satisfies (3.52) and (3.55) which contradicts the minimality of  $u$ . The proof is complete.  $\square$

Theorem 1.2 follows from Lemma 3.6.

## 4 The proof of Theorem 1.3

## 5 Basic lemmas

**Lemma 5.1.** *There exist positive constants  $c, q^*$  such that*

$$(5.1) \quad W''(q) \geq c^2, \text{ for } q \in (-q^*, q^*);$$

$$(5.2) \quad \begin{aligned} W(q) &\geq \tilde{W}(q_0, q) := W(q_0) + W'(q_0)(q - q_0), \\ &\text{for } (q_0, q) \in (0, q^*) \times (q_0, q^*] \cup (-q^*, 0) \times [-q^*, q_0); \end{aligned}$$

$$(5.3) \quad \text{sign}(q)W'(q) \geq \text{sign}(q)c^2 q \geq 0, \quad \text{for } q \in (-q^*, q^*);$$

*Proof.* The inequality (5.1) follows immediately from hypothesis (iii). Now, the convexity of  $W$  in  $(-q^*, q^*)$  implies (5.2).

To prove (5.3) note that, for  $q \in (0, q^*)$ ,

$$W'(q) = \int_0^q W''(t)dt \geq c^2 q.$$

Analogously, for  $q \in (-q^*, 0)$ ,

$$W'(q) = - \int_q^0 W''(t)dt \leq c^2 q.$$

□

By reducing the value of  $q^*$  if necessary, we can also assume

$$(5.4) \quad W(q^* \cdot \text{sign} q) \leq W(q) \leq \overline{W}, \quad \text{for } |q| \in [q^*, M_0],$$

where  $\overline{W} > 0$  is a suitable constant. This follows from assumption (iii) and (1.12).

All the arguments that follow have a local character. Therefore, without loss of generality, in the remaining part of the proof we can assume that  $\Omega$  is bounded.

**Lemma 5.2.** *Assume  $R > 0$  and  $B_{x_0, R} \subset \Omega$  and let  $\varphi : B_{x_0, R} \rightarrow \mathbb{R}$  be the solution of*

$$(5.5) \quad \begin{cases} \Delta \varphi = c^2 \varphi, & \text{in } B_{x_0, R}, \\ \varphi = \bar{q}, & \text{on } \partial B_{x_0, R}, \end{cases}$$

where  $\bar{q} \in (0, q^*]$ . Assume that  $u \in W^{1,2}(\Omega)$  is a continuous map such that

$$(5.6) \quad |u| \leq \bar{q}, \quad \text{for } x \in \overline{B}_{x_0, R}.$$

Then there exists a map  $v \in W^{1,2}(\Omega)$  that satisfies:

$$(5.7) \quad \begin{aligned} v &= u, & \text{for } x \in \Omega \setminus B_{x_0, R}, \\ |v| &\leq \varphi, & \text{for } x \in \overline{B}_{x_0, R} \end{aligned}$$

and

$$(5.8) \quad \begin{aligned} J_\Omega(u) - J_\Omega(v) &= J_{B_{x_0, R}}(u) - J_{B_{x_0, R}}(v) \\ &\geq \int_{B_{x_0, R} \cap \{u > \varphi\}} (W(u) - W(\varphi^u) - W'(\varphi^u)(u - \varphi^u)) dx, \end{aligned}$$

where  $\varphi^u = \text{sign}(u)\varphi$ .

*Proof.* Let  $b > 0$  be a number such that  $b \leq \min_{x \in B_{x_0, R}} \varphi$ . Since  $u$  is continuous the set  $A_b := \{x \in B_{x_0, R} : u > b\}$  is open and there exists a function  $\rho^+ \in W^{1,2}(A_b)$  that minimizes the functional  $J_{A_b}(p) = \int_{A_b} (\frac{1}{2} |\nabla p|^2 + W(p)) dx$  in the class of functions that satisfy the Dirichlet condition  $p = u$  on  $\partial A_b$ . Since  $\frac{|\rho^+| + \rho^+}{2}$  is also a minimizer we have  $\rho^+ \geq 0$ . We also have  $\rho^+ \leq \bar{q}$ . This follows from (5.3) and (5.4) which imply that  $\min\{\rho^+, \bar{q}\}$  is also a minimizer. The map  $\rho^+$  satisfies the variational equation

$$(5.9) \quad \int_{A_b} (\langle \nabla \rho^+, \nabla \eta \rangle + W'(\rho^+) \eta) dx = 0,$$

for all  $\eta \in W_0^{1,2}(A_b) \cap L^\infty(A_b)$ . In particular, if we define  $A_b^* := \{x \in A_b : \rho^+ > \varphi\}$ , we have

$$(5.10) \quad \int_{A_b^*} (\langle \nabla \rho^+, \nabla \eta \rangle + W'(\rho^+) \eta) dx = 0,$$

for all  $\eta \in W_0^{1,2}(A_b) \cap L^\infty(A_b)$  that vanish on  $A_b \setminus A_b^*$ .

If we take  $\eta = (\rho^+ - \varphi)^+$  in (5.10) and use (5.3) we get

$$(5.11) \quad \int_{A_b^*} (\langle \nabla \rho^+, \nabla(\rho^+ - \varphi) \rangle + c^2 \rho^+ (\rho^+ - \varphi)) dx \leq 0,$$

This inequality and

$$(5.12) \quad \int_{A_b^*} (\langle \nabla \varphi, \nabla(\rho^+ - \varphi) \rangle + c^2 \varphi (\rho^+ - \varphi)) dx = 0,$$

that follows from (5.5), imply

$$(5.13) \quad \int_{A_b^*} (|\nabla(\rho^+ - \varphi)|^2 + c^2 (\rho^+ - \varphi)^2) dx \leq 0.$$

That is  $\mathcal{H}^n(A_b^*) = 0$  which, together with  $\rho^+ \leq \varphi$  on  $A_b \setminus A_b^*$ , shows that

$$(5.14) \quad \rho^+ \leq \varphi, \text{ for } x \in A_b.$$

If we set  $A_b^- := \{x \in B_{x_0, R} : u < -b\}$  and  $\rho^- \in W^{1,2}(A_b^-)$  is a minimizer of  $J_{A_b^-}$  in the set of  $W^{1,2}(A_b^-)$  maps that have the same trace of  $u$  on  $\partial A_b^-$ , the argument above can be applied to  $\rho^-$  to obtain

$$(5.15) \quad \rho^- \geq -\varphi, \text{ for } x \in A_b^-.$$

Let  $v \in W^{1,2}(\Omega)$  be the map defined by setting

$$(5.16) \quad v = \begin{cases} u, & \text{for } x \in \Omega \setminus A_b \cup A_b^-, \\ \min\{u, \rho^+\}, & \text{for } x \in A_b, \\ \max\{u, \rho^-\}, & \text{for } x \in A_b^-, \end{cases}$$

This definition implies (5.7). Moreover we have

$$(5.17) \quad \begin{aligned} J_\Omega(u) - J_\Omega(v) &= J_{A_b \cup A_b^-}(u) - J_{A_b \cup A_b^-}(v) \\ &= J_{A_b \cap \{\rho^+ < u\}}(u) - J_{A_b \cap \{\rho^+ < u\}}(\rho^+) \\ &\quad + J_{A_b^- \cap \{\rho^- > u\}}(u) - J_{A_b^- \cap \{\rho^- > u\}}(\rho^-). \end{aligned}$$

From (5.9) with  $\eta = (u - \rho^+)^+$  it follows

$$(5.18) \quad \int_{A_b \cap \{\rho^+ < u\}} \langle \nabla \rho^+, \nabla(u - \rho^+) \rangle = - \int_{A_b \cap \{\rho^+ < u\}} W'(\rho^+) (u - \rho^+) dx.$$

This and the identity

$$(5.19) \quad \frac{1}{2} (|\nabla u|^2 - |\nabla \rho^+|^2) = \frac{1}{2} |\nabla u - \nabla \rho^+|^2 + \langle \nabla \rho^+, \nabla(u - \rho^+) \rangle,$$

imply

$$\begin{aligned}
(5.20) \quad & J_{A_b \cap \{\rho^+ < u\}}(u) - J_{A_b \cap \{\rho^+ < u\}}(\rho^+) \\
&= \int_{A_b \cap \{\rho^+ < u\}} \left( \frac{1}{2} |\nabla u - \nabla \rho^+|^2 + \langle \nabla \rho^+, \nabla(u - \rho^+) \rangle + W(u) - W(\rho^+) \right) dx \\
&\geq \int_{A_b \cap \{\rho^+ < u\}} (W(u) - W(\rho^+) - W'(\rho^+)(u - \rho^+)) dx \\
&\geq \int_{A_b \cap \{\varphi < u\}} (W(u) - W(\varphi) - W'(\varphi)(u - \varphi)) dx,
\end{aligned}$$

where we have used  $A_b \cap \{\varphi < u\} \subset A_b \cap \{\rho^+ < u\}$  and the fact that the function  $\tilde{W}(\cdot, u)$  defined in (5.2) is increasing on  $(0, u)$ . In the same way one proves

$$\begin{aligned}
(5.21) \quad & J_{A_b^- \cap \{\rho^- > u\}}(u) - J_{A_b^- \cap \{\rho^- > u\}}(\rho^-) \\
&\geq \int_{A_b^- \cap \{\varphi > u\}} (W(u) - W(-\varphi) - W'(-\varphi)(u + \varphi)) dx.
\end{aligned}$$

This inequality and (5.20) imply (5.8).  $\square$

Given  $\bar{q} \in (0, q^*)$  define

$$(5.22) \quad \bar{R} = \frac{q^* - \bar{q}}{M_0}$$

where  $M_0$  is the constant in (1.12). For later reference we quote

**Corollary 5.3.** *Let  $\lambda > 0$  be fixed, assume that  $R > \bar{R}$  is such that  $B_{x_0, R+\lambda/2} \subset \Omega$  and let  $u \in W^{1,2}(\Omega)$  a continuous map that satisfies the condition*

$$(5.23) \quad |u| \leq \bar{q}, \text{ for } x \in \partial B_{x_0, R+\lambda/2}.$$

*Then, there exist a constant  $k > 0$  independent of  $R > \bar{R}$  and a map  $v \in W^{1,2}(\Omega)$  such that*

$$(5.24) \quad v = u, \text{ on } \Omega \setminus B_{x_0, R+\lambda/2},$$

$$J_\Omega(u) - J_\Omega(v) = J_{B_{x_0, R+\lambda/2}}(u) - J_{B_{x_0, R+\lambda/2}}(v) \geq k \mathcal{H}^n(A_{\bar{q}} \cap B_{x_0, R}),$$

where  $A_{\bar{q}} := \{x \in \Omega : |u| > \bar{q}\}$ .

*Proof.* Let  $\hat{u} \in W^{1,2}(\Omega)$  be defined by

$$(5.25) \quad \hat{u} = \begin{cases} \bar{q}, & \text{on } B_{x_0, R+\lambda/2} \cap \{u > \bar{q}\}, \\ -\bar{q}, & \text{on } B_{x_0, R+\lambda/2} \cap \{u < -\bar{q}\}, \\ u, & \text{otherwise.} \end{cases}$$



Then, using also (5.4), we have

$$(5.26) \quad J_\Omega(u) - J_\Omega(\hat{u}) = \int_{B_{x_0, R+\lambda/2} \cap \{u > \bar{q}\}} \left( \frac{1}{2} |\nabla u|^2 + W(u) - W(\bar{q}) \right) dx \\ + \int_{B_{x_0, R+\lambda/2} \cap \{u < -\bar{q}\}} \left( \frac{1}{2} |\nabla u|^2 + W(u) - W(-\bar{q}) \right) dx \geq 0.$$

The map  $\hat{u}$  satisfies the assumptions of Lemma 5.2. Therefore if we let  $v$  be the map associated to  $\hat{u}$  by Lemma 5.2 (for  $R + \lambda/2$ ), from (5.26) and (5.8) we obtain

$$(5.27) \quad J_\Omega(u) - J_\Omega(v) = J_{B_{x_0, R+\lambda/2}}(u) - J_{B_{x_0, R+\lambda/2}}(v) \\ \geq J_{B_{x_0, R+\lambda/2}}(\hat{u}) - J_{B_{x_0, R+\lambda/2}}(v) \\ \geq \int_{A_{\bar{q}} \cap B_{x_0, R+\lambda/2}} (W(\hat{u}) - W(\varphi^{\hat{u}}) - W'(\varphi^{\hat{u}})(\hat{u} - \varphi^{\hat{u}})) dx,$$

where we have also used  $A_{\bar{q}} \cap B_{x_0, R+\lambda/2} \subset B_{x_0, R+\lambda/2} \cap \{|\hat{u}| > \varphi\}$ .

We have  $\varphi(x) = \phi(|x - x_0|, R + \lambda/2)$  with  $\phi(\cdot, R + \lambda/2) : [0, R + \lambda/2] \rightarrow \mathbb{R}$  a positive function which is strictly increasing in  $(0, R + \lambda/2]$ . Moreover we have  $\phi(R + \lambda/2, R + \lambda/2) = \bar{q}$  and

$$(5.28) \quad R_1 < R_2 \Rightarrow \phi(R_1 - \lambda, R_1) > \phi(R_2 - \lambda, R_2).$$

Note that  $x \in B_{x_0, R}$  implies  $\varphi(x) \leq \phi(R, R + \lambda/2)$ . Therefore for  $x$  in the subset of  $A_{\bar{q}} \cap B_{x_0, R}$  where  $u > \varphi$  we have

$$(5.29) \quad W(\bar{q}) - W(\varphi) - W'(\varphi)(\bar{q} - \varphi) = \int_{\varphi}^{\bar{q}} (W'(q) - W'(\varphi)) dq \\ \geq c^2 \int_{\varphi}^{\bar{q}} (q - \varphi) dq = \frac{1}{2} c^2 (\bar{q} - \varphi)^2 \geq \frac{1}{2} c^2 (\phi(R + \lambda/2, R + \lambda/2) - \phi(R, R + \lambda/2))^2,$$

where we have also used (5.1). In a similar way we derive the estimate

$$(5.30) \quad W(-\bar{q}) - W(-\varphi) - W'(-\varphi)(-\bar{q} + \varphi) = \int_{-\varphi}^{-\bar{q}} (W'(q) - W'(-\varphi)) dq \\ \geq -c^2 \int_{-\bar{q}}^{-\varphi} (q + \varphi) dq = \frac{1}{2} c^2 (\bar{q} - \varphi)^2 \geq \frac{1}{2} c^2 (\phi(R + \lambda/2, R + \lambda/2) - \phi(R, R + \lambda/2))^2,$$

valid in the subset of  $A_{\bar{q}} \cap B_{x_0, R}$  where  $u < -\varphi$ . The corollary follows from this and (5.29), from (5.27) and from the fact that, by (5.28), the last expression in (5.29) and (5.30) is increasing with  $R$ . Therefore we can assume

$$(5.31) \quad k = \frac{1}{2} c^2 (\phi(\bar{R} + \lambda/2, \bar{R} + \lambda/2) - \phi(\bar{R}, \bar{R} + \lambda/2))^2.$$

□

**Lemma 5.4.** *Let  $u \in W^{1,2}(\Omega)$  be a local minimizer as in Theorem 1.3. Let  $\lambda > 0$  be fixed and assume that  $B_{x_0, R+\lambda} \subset \Omega$  for some  $R > \bar{R}$ . Assume*

$$(5.32) \quad A_{\bar{q}} \cap B_{x_0, R} \neq \emptyset,$$

and let  $S = A_{\bar{q}} \cap (B_{x_0, R+\lambda} \setminus \overline{B_{x_0, R}})$ . Then, there exist a constant  $K > 0$  independent of  $R > \bar{R}$  and a continuous map  $v \in W^{1,2}(\Omega)$  that satisfies

$$(5.33) \quad \begin{cases} v = u, & \text{for } x \in \Omega \setminus S, \\ \text{sign}(u)v > \bar{q}, & \text{for } x \in A_{\bar{q}} \cap B_{x_0, R+\frac{\lambda}{2}}, \\ \text{sign}(u)v = \bar{q}, & \text{for } x \in \partial(A_{\bar{q}} \cap B_{x_0, R+\frac{\lambda}{2}}), \end{cases}$$

and

$$(5.34) \quad J_{\Omega}(v) - J_{\Omega}(u) = J_S(v) - J_S(u) \leq K\mathcal{H}^n(S).$$

*Proof.* From Corollary 5.3 and the minimality of  $u$  we necessarily have  $A_{\bar{q}} \cap \partial B_{x_0, R+\frac{\lambda}{2}} \neq \emptyset$ . Indeed, if on the contrary  $A_{\bar{q}} \cap \partial B_{x_0, R+\frac{\lambda}{2}} = \emptyset$ , then  $|u| \leq \bar{q}$  on  $\partial B_{x_0, R+\frac{\lambda}{2}}$ . Therefore, applying Corollary 5.3 to  $u$  on  $B_{x_0, R+\frac{\lambda}{2}}$ , we could find  $v$  satisfying

$$J_{\Omega}(u) - J_{\Omega}(v) = J_{B_{x_0, R+\frac{\lambda}{2}}}(u) - J_{B_{x_0, R+\frac{\lambda}{2}}}(v) \geq k\mathcal{H}^n(A_{\bar{q}} \cap B_{x_0, R}).$$

From (5.32), this is in contradiction with the minimality of  $u$ .

Let  $v \in W^{1,2}(\Omega)$  be defined by  $v = u$  for  $x \notin S$  and by

$$(5.35) \quad v = (1 - |1 - 2\frac{r-R}{\lambda}|)\text{sign}(u)\bar{q} + |1 - 2\frac{r-R}{\lambda}|u, \text{ for } x \in S,$$

where  $r = |x - x_0|$ . From this definition and (1.12) it follows

$$(5.36) \quad \begin{aligned} \bar{q} < \text{sign}(u)v \leq |u| \leq M_0, & \text{ for } x \in S \setminus \partial B_{x_0, R+\frac{\lambda}{2}}, \\ v = \text{sign}(u)\bar{q}, & \text{ for } x \in S \cap \partial B_{x_0, R+\frac{\lambda}{2}}. \end{aligned}$$

Moreover, it is easy to verify that  $v = u$  on  $\partial S$ . Then,  $v$  is continuous and satisfies (5.33). From (5.35) we also obtain

$$(5.37) \quad \nabla v = \left|1 - 2\frac{r-R}{\lambda}\right| \nabla u + \frac{2}{\lambda}(u - \text{sign}(u)\bar{q})\nu, \text{ for } x \in S,$$

where  $\nu = -\text{sign}(1 - 2\frac{r-R}{\lambda})\frac{x-x_0}{r}$ . From (5.37), (5.36) and (1.12) it follows

$$(5.38) \quad \begin{aligned} & \frac{1}{2}(|\nabla v|^2 - |\nabla u|^2) + W(v) - W(u) \\ & \leq \frac{1}{2}(|\nabla u| + \frac{2}{\lambda}|u - \text{sign}(u)\bar{q}|)^2 + \overline{W} - W(\text{sign}(u)\bar{q}) \\ & \leq \frac{1}{2}(M_0 + \frac{2}{\lambda}(M_0 - \bar{q}))^2 + \overline{W}, \text{ for } x \in S, \end{aligned}$$

where  $\overline{W}$  is the constant in Lemma 5.1. The estimate (5.38) concludes the proof with  $K$  given by the last expression in (5.38).  $\square$

**Proposition 5.5.** *Let  $\bar{q} \in (0, q^*)$ ,  $\lambda > 0$  and  $\bar{R} = \frac{q^* - \bar{q}}{M_0}$  as before. There exists  $j_m \in \mathbb{N}$  such that, if  $R_0 = \bar{R} + (j_m + 1)\lambda$ , then a local minimizer  $u$  satisfies*

$$(5.39) \quad x \in \Omega, \quad d(x, \partial\Omega) \geq R_0 \quad \Rightarrow \quad |u| < q^*.$$

Moreover the number  $j_m$  depends only on  $\bar{q}$ ,  $\lambda$  and the constants  $k$ ,  $K$  in Corollary 5.3 and Lemma 5.4.

*Proof.* Suppose that  $|u(x_0)| \geq q^*$  for some  $x_0 \in \Omega$ . Then, from (1.12),

$$|u(x)| > \bar{q}, \quad \forall x \in B_{x_0, \bar{R}}.$$

Therefore, if  $d(x_0, \partial\Omega) \geq \bar{R}$ , (1.12) implies

$$(5.40) \quad \mathcal{H}^n(A_{\bar{q}} \cap B_{x_0, \bar{R}}) = \mathcal{H}^n(B_{x_0, \bar{R}}) := \sigma_0.$$

Now, set

$$(5.41) \quad \sigma_j := \mathcal{H}^n(A_{\bar{q}} \cap B_{x_0, \bar{R}+j\lambda}),$$

for each  $j \in \mathbb{N}$  such that  $d(x_0, \partial\Omega) \geq \bar{R} + (j+1)\lambda$ .

Let  $v_j^1, v_j^2 \in W^{1,2}(\Omega)$  be the maps defined as follows:

$v_j^1$  is the map  $v$  defined in Lemma 5.4 for  $B_{x_0, R+\lambda}$  with  $R = \bar{R} + j\lambda$ .

$v_j^2$  is the map  $v$  given by Corollary 5.3 when  $u = v_j^1$  and  $R = \bar{R} + j\lambda$ .

From these definitions, Corollary 5.3 and Lemma 5.4, we deduce

$$(5.42) \quad \begin{aligned} J_\Omega(u) - J_\Omega(v_j^1) &\geq -K(\sigma_{j+1} - \sigma_j), \\ J_\Omega(v_j^1) - J_\Omega(v_j^2) &\geq k\mathcal{H}^n(A_{\bar{q}} \cap B_{x_0, \bar{R}+j\lambda}) = k\sigma_j. \end{aligned}$$

By adding these inequalities and using the minimality of  $u$  we obtain

$$(5.43) \quad 0 \geq J_\Omega(u) - J_\Omega(v_j^2) \geq k\sigma_j - K(\sigma_{j+1} - \sigma_j)$$

and therefore,

$$(5.44) \quad \begin{aligned} \left(1 + \frac{k}{K}\right)\sigma_{j-1} \leq \sigma_j &\leq \frac{K}{k}(\sigma_{j+1} - \sigma_j), \quad j \in \mathbb{N}, \\ \Rightarrow \left(1 + \frac{k}{K}\right)^{j-1} \sigma_0 \leq \sigma_j &\leq \omega \frac{K}{k} \left( (\bar{R} + (j+1)\lambda)^n - (\bar{R} + j\lambda)^n \right), \quad j \in \mathbb{N}. \end{aligned}$$

where  $\omega$  is the measure of the unit ball in  $\mathbb{R}^n$ . For  $j$  sufficiently large the last inequality is not satisfied and this contradicts the minimality of  $u$ . We denote  $j_m$  the minimum value of  $j$  such that (5.44) is violated. Then, (5.39) follows with  $R_0 = \bar{R} + (j_m + 1)\lambda$ .  $\square$

The existence of the map  $(0, q^*] \ni q \rightarrow R(q)$  follows from the fact that all the above arguments can be repeated with a generic  $q \in (0, q^*)$  in place of  $q^*$ . We can obviously assume that  $R(q)$  is decreasing and, by modifying it if necessary, we can also assume that it is strictly decreasing and continuous.

For completing the proof of Theorem 1.3 it remains to prove the estimate (1.14). Proposition 5.5 and in particular (5.39) imply that we can apply Lemma 5.2 to  $u$  and the ball  $B_{x,R}$  for each  $x \in \Omega$  such that  $d(x, \partial\Omega) = R_0 + R$  with  $R \geq R_0$ . Therefore we obtain

$$(5.45) \quad |u(x)| \leq \phi(0, R).$$

We also have (see [5]) that

$$(5.46) \quad \phi(0, R) \leq q^* e^{-k_0 R} = q^* e^{k_0 R_0} e^{-k_0 d(x, \partial\Omega)},$$

for some  $k_0 > 0$  independent of  $R \in [\bar{R}, +\infty)$ . From (5.45), (5.46) we obtain

$$(5.47) \quad |u(x)| \leq q(R) \leq K_0 e^{-k_0 d(x, \partial\Omega)}, \quad \text{for } d(x, \partial\Omega) \geq 2R_0,$$

This concludes the proof of Theorem 1.3.

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